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Progress Report No. 5

STUDY OF MULTI-STATE PN SEQUENCES AND THEIR APPLICATION TO COMMUNICATION SYSTEMS

CONTRACT N00014-75-C-1040

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I. Introduction

This document is the fifth progress report submitted each six weeks in accordance with the requirements of contract N00014-75-C-1040 and contains a summary of the work accomplished on the subject contract during the reporting period 1 January 1976 through 15 February 1976. The work outlined in the Contract Work Statement is proceeding on schedule and no unanticipated difficulties or problem areas have been encountered.

II. Work Accomplished During the Current Reporting Period

Substantial progress was made during the current reporting period which was devoted to the following areas:

- Further correlation properties of four-state sequences.
- Implementation of four-state sequences in systems using correlation detectors.
- Spectral properties of four-state sequences.

The cross-correlation function of any two maximal four-state sequences was determined in terms of the cross-correlation function of the binary sequences used to represent them. Two general formulas were obtained, each of which applies depending on whether the four-state sequences being correlated are of the same or opposite category. It was shown that the cross-correlation function of two maximal four-state sequences of opposite category is real valued.

The mean square value of the cross-correlation function of any two maximal four-state sequences has been determined. For two maximal four-state sequences of opposite category the mean square value of their cross-correlation function is found to be less than that for binary maximal sequences of the same period. We thus conclude that the four-state sequences will perform better with respect to their cross-correlation properties than binary sequences of the same period.

The role of the correlation function of two four-state sequences as previously defined in this study was examined from an implementation point of view. Specifically, the output of a correlation detector receiving a four-state encoded signal and using a four-state encoded local reference was computed and the dependence of this output on the correlation function of the corresponding sequences was determined.

The spectral properties of signals encoded with four-state sequences were studied and a general formula for the determination of their spectra was developed. This formula was applied to determine the power density spectrum of maximal four-state sequences.

III. Meetings

A presentation describing the results to date on the subject study was made to Naval Research Laboratories personnel by Dr. Robert Gold, consultant to Collins Radio Group, on 14 January 1976. Those properties of four-state sequences having potential applications to coded communication systems were emphasized. Mr. Jim Allen of Naval Research Laboratories suggested subsequent presentations be arranged in order to acquaint potential users with the techniques being developed.

IV. Personnel

Personnel expending effort on the subject contract during the current reporting period: Dr. Robert Gold, consultant, Collins Radio Group.

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V. Plans For Next Reporting Period

In accordance with the contract work statement, a study of the correlation properties of four-state Gold families will be initiated.

VI. Technical Appendices

Appendix E included in this report contains the technical details of the results described above.

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APPENDIX E

CORRELATION AND SPECTRAL PROPERTIES OF FOUR-STATE SEQUENCES

E.1 GENERAL EXPRESSION FOR CROSS-CORRELATION OF MAXIMAL FOUR-STATE SEQUENCES.

If a is four-state maximal sequence generated by the primitive four-state polynomial $f(x) \in GF(4)[x]$ then we have shown in a previous theorem that the four-state sequence a can be represented as the interleaving of two binary sequences generated by the binary polynomial $f(x)\,\overline{f(x)}$. In particular, with respect to the basis $\{\beta,\beta^2\}$ we have

$$a = (a_0, a_{P/3}) \text{ for } f(x) \text{ of category one } \left(f(0) = \beta\right) \text{ and}$$

$$a = (a_0, a_{\frac{2P}{3}}) \text{ for } f(x) \text{ of category two } \left(f(0) = \beta^2\right) \text{ when}$$

$$a_0 = \frac{P_1\left(\overline{f(x)}\right)}{f(x)\,\overline{f(x)}}$$

In this section we determine the cross-correlation function of any two maximal four-state sequences a and b in terms of the cross-correlation functions of the binary maximal sequences used to represent them.

<u>Theorem:</u> Let a and b be two maximal four-state sequences

$a \in V\left(f(x)\right);\ f(x) \in GF(4)[x]$.

$b \in V\left(g(x)\right);\ g(x) \in GF(4)[x]$.

Let $a_0 = \dfrac{P_1\left(\overline{f(x)}\right)}{f(x)\,\overline{f(x)}}$ and $b_0 = \dfrac{P_1\left(\overline{g(x)}\right)}{f(x)\,\overline{f(x)}}$

with respect to the basis (β,β^2). Then if a and b are of the same category one

$$\mathrm{Re}\ \theta(a,b)(\tau)=\theta(a_0,b_0)\ (\tau)$$

$$\mathrm{Im}\ \theta(a,b)(\tau)=\frac{\theta(a_0,b_{2P/3})(\tau)-\theta(a_0,b_{P/3})(\tau)}{2}$$

if a and b are of different category a category one b category 2

$$\mathrm{Re}\ \theta(a,b)(\tau)=\frac{\theta(a_0,b_0)(\tau)+\theta(a_0,b_{P/3})}{2}$$

$$\mathrm{Im}\ \theta(a,b)(\tau)=0$$

Proof: By a previous theorem we know that if a and b are any two four-state sequences then

$$\theta(a,b)(\tau)=\left[\frac{\theta(a_1,b_1)(\tau)+\theta(a_2,b_2)(\tau)}{2}\right]+j\left[\frac{\theta(a_2,b_1)(\tau)-\theta(a_1,b_2)(\tau)}{2}\right]$$

Where $\left.\begin{matrix}[a_1,a_2]\\ [b_1,b_2]\end{matrix}\right\}$ is the representation with respect to the basis $\{\beta,\beta^2\}$

of the four-state sequence $\begin{Bmatrix}a\\ b\end{Bmatrix}$ as the interleaving of binary sequences.

Case 1: Suppose a and b are both of category 1 then by a previous theorem we have $a=(a_0,a_{P/3})$ $b=(b_0,b_{P/3})$ with respect to the basis $\{\beta,\beta^2\}$. Thus using the above expression for $\theta(a,b)$ we have

$$\mathrm{Re}\ \theta(a,b)(\tau)=\frac{\theta(a_0,b_0)(\tau)+\theta(a_{P/3},b_{P/3})(\tau)}{2}=\theta(a_0,b_0)(\tau)$$

with respect to the basis (β,β^2). Then if a and b are of the same category one

$$\mathrm{Re}\ \theta(a,b)(\tau) = \theta(a_0,b_0)\ (\tau)$$

$$\mathrm{Im}\ \theta(a,b)(\tau) = \frac{\theta(a_0,b_{2P/3})(\tau) - \theta(a_0,b_{P/3})(\tau)}{2}$$

if a and b are of different category a category one b category 2

$$\mathrm{Re}\ \theta(a,b)(\tau) = \frac{\theta(a_0,b_0)(\tau) + \theta(a_0,b_{P/3})}{2}$$

$$\mathrm{Im}\ \theta(a,b)(\tau) = 0$$

Proof: By a previous theorem we know that if a and b are any two four-state sequences then

$$\theta(a,b)(\tau) = \left[\frac{\theta(a_1,b_1)(\tau) + \theta(a_2,b_2)(\tau)}{2}\right] + j\left[\frac{\theta(a_2,b_1)(\tau) - \theta(a_1,b_2)(\tau)}{2}\right]$$

Where $\left.\begin{matrix}[a_1,a_2]\\ [b_1,b_2]\end{matrix}\right\}$ is the representation with respect to the basis $\{\beta,\beta^2\}$

of the four-state sequence $\begin{Bmatrix}a\\ b\end{Bmatrix}$ as the interleaving of binary sequences.

Case 1: Suppose a and b are both of category 1 then by a previous theorem we have $a = (a_0,a_{P/3})$ $b = (b_0,b_{P/3})$ with respect to the basis $\{\beta,\beta^2\}$. Thus using the above expression for $\theta(a,b)$ we have

$$\mathrm{Re}\ \theta(a,b)(\tau) = \frac{\theta(a_0,b_0)(\tau) + \theta(a_{P/3},b_{P/3})(\tau)}{2} = \theta(a_0,b_0)(\tau)$$

$$\text{Im } \theta(a,b)(\tau) = \frac{\theta(a_{P/3}, b_0)(\tau) - \theta(a_0, b_{P/3})(\tau)}{2}$$

$$= \frac{\theta(a_0, b_{2P/3})(\tau) - \theta(a_0, b_{P/3})(\tau)}{2}$$

Case 2: Suppose a is of category one and b is of category two. Then by previous theorem we have

$$a = \left[a_0, a_{P/3}\right] \qquad b = \left[b_0, b_{2P/3}\right]$$

with respect to the basis (β, β^2). Thus, using the above expression for $\theta(a,b)$ we have

$$\text{Re } \left[\theta(a,b)(\tau)\right] = \frac{\theta(a_0, b_0)(\tau) + \theta(a_{P/3}, b_{2P/3})(\tau)}{2}$$

$$= \frac{\theta(a_0, b_0)(\tau) + \theta(a_0, b_{P/3})(\tau)}{2}$$

$$\text{Im } \left[\theta(a,b)(\tau)\right] = \frac{\theta(a_{P/3}, b_0)(\tau) - \theta(a_0, b_{2P/3})(\tau)}{2} = 0$$

Example: In what follows we illustrate the results of the above theorem.

(1) We select two maximal four-state polynomials from our tables of degree 2. These polynomials are listed in the tables as 1 All and 7 AA1 and they represent the second degree polynomials $f(x)=\beta+x+x^2$ and $g(x)= \beta+\beta x+x^2$ respectively. We note that since $f(0)=g(0)=\beta$ these polynomials are of the same category.

The polynomial $\overline{g(x)} = h(x) = \beta^2+\beta^2x+x^2$ is of the opposite category to g(x) and f(x).

(2) The characteristic sequences of $V(\beta+x+x^2)$, $V(\beta+\beta x+x^2)$ and $V(\beta^2+\beta^2x+x^2)$ are

$$\begin{array}{lccccccccccccccccc}
 & & & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
a = \dfrac{(xf)'+f}{f} & = & \dfrac{x}{\beta+x+x^2} = & 0 & \beta^2 & \beta & \beta^2 & \beta^2 & 0 & \beta & 1 & \beta & \beta & 0 & 1 & \beta^2 & 1 & 1 \\
b = \dfrac{(xg)'+g}{g} & = & \dfrac{\beta x}{\beta+\beta x+x^2} = & 0 & 1 & 1 & \beta & 1 & 0 & \beta^2 & \beta^2 & 1 & \beta^2 & 0 & \beta & \beta & \beta^2 & \beta \\
c = \dfrac{(xh)'+h}{h} & = & \dfrac{\beta^2 x}{\beta^2+\beta^2x+x^2} = & 0 & 1 & 1 & \beta^2 & 1 & 0 & \beta & \beta & 0 & \beta & 0 & \beta^2 & \beta^2 & \beta & \beta^2
\end{array}$$

(3) With respect to the basis (β,β^2) these four-state maximal sequences are represented as the interleaved sequences

$a = \left[a_0, a_5\right]$ $b = \left[b_0, b_5\right]$ (Category 1) $c = \left[c_0, c_{10}\right]$ (Category 2)

$$\begin{aligned}
a &= 0\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 1 \\
b &= 0\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0 \\
c &= 0\ 0\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 1
\end{aligned}$$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
$a_0=$	0	0	1	0	0	0	1	1	1	1	0	1	0	1	1	$\varepsilon V(1+x+x^4)$
$b_0=$	0	1	1	1	1	0	0	0	1	0	0	1	1	0	1	$\varepsilon V(1+x^3+x^4)$
$c_0=$	0	1	1	0	1	0	1	1	1	1	0	0	0	1	0	$\varepsilon V(1+x^3+x^4)$

(4) The cross-correlation function of the four-state maximal sequences a and b which are both of the first category and sequences a and c where sequence c is of the second category may be expressed in terms of the cross-correlation function of the maximal binary sequences a_0, b_0 and a_0, c_0 respectively. The required binary cross-correlation functions are readily computed to be:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\theta(a_0,b_0)(\tau)$ =	-1	-5	-5	3	-5	7	3	-1	-5	3	7	-1	3	-1	-1
$\theta(a_0,b_{10})(\tau)$ =	7	-1	3	-1	-1	-1	-5	-5	3	-5	7	3	-1	-5	3
$\theta(a_0,b_5)(\tau)$ =	7	3	-1	-5	3	7	-1	3	-1	-1	-1	-5	-5	3	-5
$\theta(a_0,c_0)(\tau)$ =	7	3	-1	-5	3	7	-1	3	-1	-1	-1	-5	-5	3	-5
$\theta(a_0,c_5)(\tau)$ =	7	-1	3	-1	-1	-1	-5	-5	3	-5	7	3	-1	-5	3

using the above theorem we find

$$\theta(a,b)(\tau) = \theta(a_0,b_0)(\tau) + j\,\frac{\theta(a_0,b_{10})(\tau) - \theta(a_0,b_5)}{2}$$

$$\theta(a,c)(\tau) = \frac{\theta(a_0,c_0)(\tau) + \theta(a_0,c_5)(\tau)}{2} + j\cdot 0$$

Thus we have

τ	$\theta(a,b)(\tau)$	$\theta(a,c)(\tau)$
0	-1 + 0j	7 + 0j
1	-5 - 2j	1 + 0j
2	-5 + 2j	1 + 0j
3	3 + 2j	-3 + 0j
4	-5 - 2j	1 + 0j
5	7 - 4j	3 + 0j
6	3 - 2j	-3 + 0j

7	-1 - 4j	-1 + 0j
8	-5 + 2j	1 + 0j
9	3 - 2j	-3 + 0j
10	7 + 4j	3 + 0j
11	-1 + 4j	-1 + 0j
12	3 + 2j	-3 + 0j
13	-1 - 4j	-1 + 0j
14	-1 + 4j	-1 + 0j

These correlation functions may be verified by computing $\theta(a,b)$ and $\theta(a,c)$ directly from the four-state sequences.

The above result shows that the cross-correlation function $\theta(a,c)$ of any two maximal four-state sequences a and c of opposite category is real valued. Furthermore, the upper bound of $\left|\theta(a,c)\right|$ is less than or equal to the upper bound of $\theta(a_0,c_0)$ where a_0 and c_0 are the corresponding binary sequences whose interleaving represents a and c respectively. In fact we have

$$\max_{\tau} \quad \theta(a,c)(\tau)$$

$$\max_{\tau} \quad \frac{\theta(a_0,c_0)(\tau) + \theta(a_0,c_5)(\tau)}{2}$$

$$\leq \max_{\tau} \left[\frac{\left|\theta(a_0,c_0)(\tau)\right|}{2} + \frac{\left|\theta(a_0,c_5)(\tau)\right|}{2} \right]$$

$$\leq \max_{\tau} \frac{\left|\theta(a_0,c_0)(\tau)\right|}{2} + \max_{\tau} \frac{\left|\theta(a_0,c_5)(\tau)\right|}{2}$$

$$= \max_{\tau} \left|\theta(a_0,c_0)(\tau)\right|$$

A plot of the cross-correlation function of the binary and four-state sequences of opposite category is presented in figure E-1.

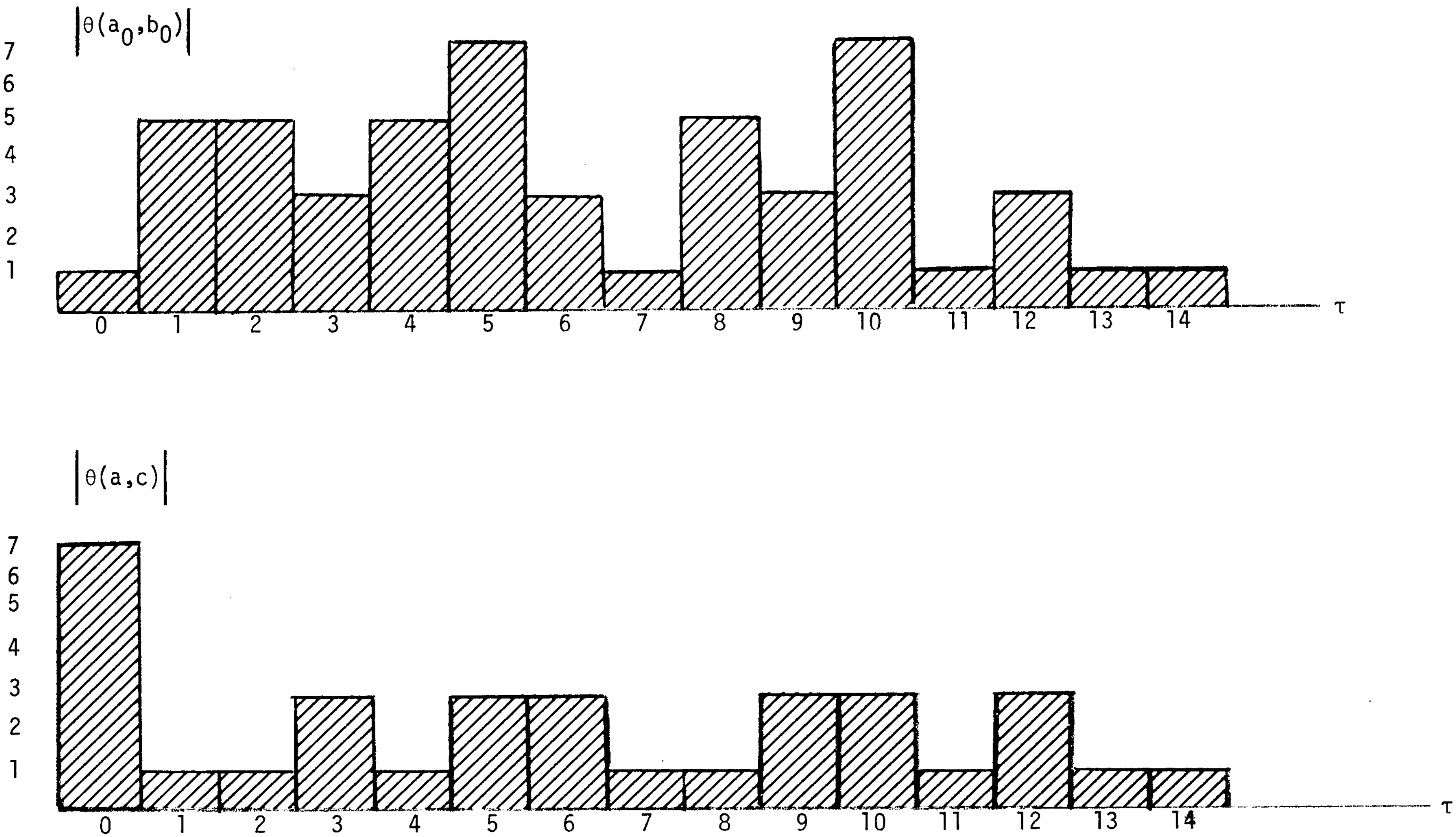

Figure E-1 : Comparison of Cross-Correlation of four-state and binary maximal sequences of opposite category.

E.2 SUM OF SQUARES OF CROSS-CORRELATION FUNCTION VALUES

In the previous section we showed that the bound on the absolute value of the cross-correlation function of two four-state maximal sequences of opposite category is less than or equal to the bound on the absolute value of the cross-correlation function of the two corresponding binary maximal sequences. We now compute the sum of the squares of the absolute values of the cross-correlation function of two maximal four-state and binary sequences and show that for maximal four-state sequences of opposite category, this sum of squares is less than the sum for the correlation function of the corresponding binary sequences.

Theorem: Let a and b be maximal four-state sequences of period $4^n-1=P$. Then

$$\sum_{\tau=0}^{P-1} \left|\theta(a,b)(\tau)\right|^2 = \begin{cases} 3\cdot 2^{4n-1} - 2^{2n} - 1 & \text{for a and b of the same category} \\ 2\cdot 2^{4n-1} - 2^{2n} - 1 & \text{for a and b binary} \\ 2^{4n-1} - 2^{2n} - 1 & \text{for a and b of the opposite category} \end{cases}$$

Proof: We have shown in a previous theorem that for any two maximal four-state sequences of period P we have

$$\sum_{\tau=0}^{P-1} \left|\theta(a,b)(\tau)\right|^2 = \sum_{\tau=0}^{P-1} \theta(a,a)(\tau)\ \overline{\theta(b,b)(\tau)}$$

For a and b four-state maximal sequences of the same category or binary, we have $\theta(a,a)(\tau) = \theta(b,b)(\tau)$ and hence

$$\sum_{\tau=0}^{P-1} \left|\theta(a,b)^2(\tau)\right| = \sum_{\tau=0}^{P-1} \left|\theta(a,a)(\tau)\right|^2$$

Using our previous evaluation of the auto correlation function of a maximal four-state sequence we have

$$\begin{aligned}\sum_{\tau=0}^{P-1} \left|\theta(a,a)(\tau)\right|^2 &= (4^n-1)^2 + 2\left(-1+j\cdot\frac{4^n}{2}\right)\left(-1-j\cdot\frac{4^n}{2}\right) + 4^n-4 \\ &= 3\cdot 2^{4n-1} - 2^{2n} - 1\end{aligned}$$

For a and b binary sequences

$$\begin{aligned}\sum_{\tau=0}^{P-1} \left|\theta(a,a)(\tau)\right|^2 &= (4^n-1)^2 + 4^n-2 \\ &= 2\cdot 2^{4n-1} - 2^{2n} - 1\end{aligned}$$

For a and b four-state maximal sequences of opposite category

$$\theta(a,a)(\tau) = \theta(b,b)(\tau) \quad \text{for} \quad \tau=0 \quad \tau \neq \frac{4^n-1}{3}, \frac{2(4^n-1)}{3}$$

$$\left.\begin{aligned}\theta(a,a)\left(\frac{4^n-1}{3}\right) &= \overline{\theta(b,b)\left(\frac{4^n-1}{3}\right)} = -1+j\,\frac{4^n}{2} \\ \theta(a,a)\left(\frac{2(4^n-1)}{3}\right) &= \overline{\theta(b,b)\left(\frac{2(4^n-1)}{3}\right)} = -1-j\,\frac{4^n}{2}\end{aligned}\right\} \quad \text{a of say first category}$$

Hence we have

$$\sum_{\tau=0}^{P-1} \left|\theta(a,b)^2(\tau)\right| = \sum_{\tau=0}^{P-1} \theta(a,a)(\tau)\ \overline{\theta(b,b)(\tau)}$$

$$= (4^n-1)^2 + \left(-1+j\ \frac{4^n}{2}\right)^2 + \left(-1-j\ \frac{4^n}{2}\right)^2 + 4^n-4$$

$$= 2^{4n-1} - 2^{2n} - 1$$

For the sequences of period $4^2-1=15$ of the previous example, we have

τ	a,b same category $\left|\theta(a,b)^2(\tau)\right|$	a_0,b_0 binary $\left|\theta(a_0,b_0)^2(\tau)\right|$	a,c opposite category $\left|\theta(a,c)^2(\tau)\right|$
0	1	1	49
1	29	25	1
2	29	25	1
3	13	9	9
4	29	25	1
5	65	49	9
6	13	9	9
7	17	1	1
8	29	25	1
9	13	9	9
10	65	49	9
11	17	1	1
12	13	9	9
13	17	1	1
14	17	1	1
	$\sum =$ 367	$\sum =$ 239	$\sum =$ 111

For $n = 2$

$$3 \cdot 2^{4n-1} - 2^{2n} - 1 = 367$$

$$2 \cdot 2^{4n-1} - 2^{2n} - 1 = 239$$

$$1 \cdot 2^{4n-1} - 2^{2n} - 1 = 111$$

E.3 ROLE OF θ IN CORRELATION DETECTION

E.3-1 ENCODING OF SIGNALS

In this section we describe the use of the four-state sequence to encode a carrier. Each of the four-states may be used to determine a phase of the carrier. Let a be the four-state sequence. Let γ be the real valued mapping of GF(4) given by

$$\gamma(0) = 0; \quad \gamma(1) = \pi; \quad \gamma(\beta) = \frac{\pi}{2}; \quad \gamma(\beta^2) = \frac{3\pi}{2}$$

The encoded carrier is then given by the equation

$$s(t) = \sum_{k=-\infty}^{\infty} \cos \omega t + \gamma\left(a(k)\right) \; f_0 \; (t - \Delta k)$$

$$\text{where } f_0(t) = 1 \qquad \text{for } 0 \leq t \leq \Delta$$
$$= 0 \qquad \text{Otherwise}$$

The encoded carrier is illustrated in Figure E-2.

Code a	β	1	1	2	•••
(a)	π/2	π	π	3π/2	•••
$\cos \omega t + \gamma\left[a(k)\right]$					•••

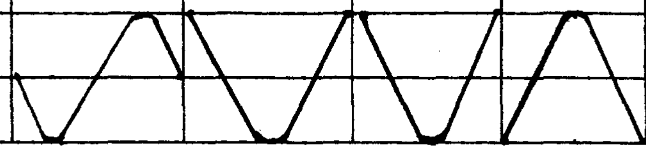

Figure E-2. Carrier Encoded with Four-State Sequence

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The above encoded carrier may also be represented as the real part of a complex signal, i.e.:

$$s(t) = \text{Re}\left[f(t)e^{i\omega t}\right] \quad \text{where} \quad f(t) = \sum_{k=-\infty}^{\infty} e^{i\gamma\left(a(k)\right)} \quad f_0(t-\Delta k)$$

We note that $\eta(x) = e^{i\gamma(x)}$ for all $x \in GF(4)$

where η is the previously defined complex valued mapping of GF(4)

$\eta(0) = 0 \qquad \eta(\beta) = i$

$\eta(1) = -1 \qquad \eta(\beta^2) = -i$

Thus the encoded carrier has the representation

$$s(t) = \text{Re}\left[f(t)\ e^{i\omega t}\right] \quad \text{where} \quad f(t) = \sum_{k=-\infty}^{\infty} \eta\left(a(k)\right)\ f_0(t-\Delta k)$$

E.3-2 COMPUTATION OF OUTPUT OF CORRELATION DETECTOR

In a spread spectrum communication system, the incoming encoded signal $s_1(t)$ is multiplied by a locally generated replica of the received signal to remove the code and recover the base-band information. In this section we describe this process for four-state sequences and indicate the importance of their correlation function.

Let the received signal be:

$$s_1(t) = \sum_{k=-\infty}^{\infty} \cos\left(\omega t + \gamma\left(a(k)\right)\right) \quad f_0(t-\Delta k) = \text{Re}\ f_1(t)e^{i\omega t}$$

Where a is four-state sequences

γ is real valued mapping of GF(4) $\quad 0 \to 0 \quad \beta \to \pi/2$

$1 \to \pi \quad \beta^2 \to 3\pi/2$

$$f_1(t) = \sum_{k=-\infty}^{\infty} \eta\big(a(k)\big) f_0\,(t-\Delta k)$$

$$\eta(x) = e^{i\gamma(x)}$$

Let the locally generated signal be identical in form to $s_1(t)$ but encoded with a possibly different four-state sequence b:

$$\text{Then } R(\tau) = \int_0^{\Delta P} s_1(t)\, s_2(t-\tau)\, dt$$

$$= \sum_{\ell=-\infty}^{\infty} \left[\frac{(\cos \omega\tau)\ \mathrm{Re}\ \theta(a,b)(\ell) - \sin \omega\tau\ \mathrm{Im}\ \theta(a,b)(\ell)}{2}\right] R(f_0,f_0)(\tau-\Delta\ell)$$

$$\text{Where } R(f_0,f_0)(\tau) = \int_{-\infty}^{\infty} f_0(s)\, f_0(s-\tau)ds$$

We prove the above result in three steps

(1) We first show that

$$R(\tau) = \int_0^{\Delta P} s_1(t)\, s_2(t-\tau)dt = \frac{1}{2}\ \mathrm{Re}\left\{e^{i\omega\tau}\int_0^{\Delta P} f_1(t)\ \overline{f_2(t-\tau)}dt\right\}$$

$$\text{where } f_1(t) = \sum_{k=-\infty}^{\infty} \eta\big(a(k)\big)\ f_0\ (t-\Delta k)$$

$$f_2(t) = \sum_{k=-\infty}^{\infty} \eta\big(b(k)\big)\ f_0\ (t-\Delta k)$$

This result shows that the correlation function of the received signal and the locally generated reference depends essentially on the correlation function of the baseband encoded wave forms

f_1 and f_2 i.e., on $\int_0^{\Delta P} f_1(t)\, \overline{f_2(t-\tau)}dt$.

(2) We next show that

$$R(f_1,f_2)(\tau) = \int_0^{\Delta P} f_1(t)\, \overline{f_2(t-\tau)}dt = \sum_{k=-\infty}^{\infty} \theta(a,b)(k)\; R(f_0,f_0)(\tau-\Delta k)$$

where $\theta(f_0,f_0)(\tau) = \int_{-\infty}^{\infty} f_0(t)\; f_0(t-\tau)dt$

This result shows that the correlation function of the encoded baseband wave forms depends essentially on the correlation function $\theta(a,b)$ of the four-state encoding sequences a and b.

(3) The final expression for the correlation function of the received and locally generated signals $s_1(t)$ and $s_2(t)$ respectively is obtained by substituting the expression for the correlation function of the baseband signals given in (2),

$$R(f_1,f_2)(\tau) = \sum_{k=-\infty}^{\infty} \theta(a,b)(\tau)\quad R(f_0,f_0)(\tau-\Delta k)$$

into the expression for the correlation function of the received and locally generated signals given in (1)

$$\int_0^{\Delta P} s_1(t)\, s_2(t-\tau)dt = \frac{1}{2}\ \mathrm{Re}\left[e^{i\omega\tau} \int_0^{\Delta P} f_1(t)\, \overline{f_2(t-\tau)}dt\right]$$

$$= \frac{1}{2}\ \mathrm{Re}\left[e^{i\omega\tau} \sum_{k=-\infty}^{\infty} \left[\mathrm{Re}\ \theta(a,b)(k) + i \quad \mathrm{Im}\ \theta(a,b)(k)\right]\ R(f_0,f_0)(t-\Delta k)\right]$$

$$= \sum_{k=-\infty}^{\infty} \left[\frac{(\cos \omega\tau)\ \mathrm{Re}\ \ \theta(a,b)(k) - (\sin \omega\tau)\ \mathrm{Im}\ \theta(a,b)(k)}{2}\right] R(f_0,f_0)(t-\Delta k)$$

Proof of result (1)

$R(\tau)$

$$\int_0^{\Delta P} s_1(t)\, s_2(t-\tau)dt$$

$$\int_0^{\Delta P} \mathrm{Re}\left[f_1(t)e^{i\omega t}\right]\ \mathrm{Re}\left[f_2(t-\tau)e^{i\omega(t-\tau)}\right]dt$$

We now use the identity $(\mathrm{Re}\ z_1)\ (\mathrm{Re}\ z_2) = \dfrac{\mathrm{Re}(z_1 \cdot z_2) + \mathrm{Re}(z_1 \cdot \overline{z}_2)}{2}$

$$\frac{1}{2}\int_{0}^{\Delta P}\left[\mathrm{Re}\quad f_1(t)\ f_2(t-\tau)e^{i\omega(2t-\tau)}\quad + \quad \mathrm{Re}\ f_1(t)\ \overline{f_2(t-\tau)e^{i\omega\tau}}\right]dt$$

Since the integral of the first term is zero, we have:

$$\frac{1}{2}\ \mathrm{Re}\left[e^{i\omega\tau}\int_{0}^{\Delta P}\quad f_1(t)\ \overline{f_2(t-\tau)}dt\right]$$

Proof of result (2)

$R\ (f_1,f_2)(\tau)$

$$\int_{0}^{\Delta P}\quad f_1(t)\ \overline{f_2(t-\tau)}\ dt$$

$$\int_{0}^{\Delta P}\left[\sum_{k=-\infty}^{\infty}\quad \eta\ \big(a(k)\big)\, f_0(t-\Delta k)\right]\left[\sum_{j=-\infty}^{\infty}\quad \overline{\eta\big(b(k)\big)}\ f_0(t-\tau-\Delta j)\right]\ dt$$

$$\sum_{k=-\infty}^{\infty}\quad\sum_{j=-\infty}^{\infty}\eta\big(a(k)\big)\,\overline{\eta\big(b(k)\big)}\int_{0}^{\Delta P}\quad f_0(t-\Delta k)\ \ f_0(t-\tau-\Delta j)\ dt$$

$$\sum_{j=-\infty}^{\infty}\quad\sum_{k=0}^{P-1}\eta\big(a(k)\big)\overline{\eta\big(b(j)\big)}\int_{\Delta k}^{\Delta(k+1)}\quad f_0(t-\Delta k)\cdot f_0(t-\tau-\Delta j)\ dt$$

Let $s = t-\Delta k$ so that $t-\tau-\Delta j = s-\tau + \Delta(k-j)$

$$\sum_{j=-\infty}^{\infty} \sum_{k=0}^{P-1} \eta\big(a(k)\big) \overline{\eta\big(b(j)\big)} \int_{0}^{\Delta} f_0(s)\, f_0\big(s-\tau + \Delta(k-j)\big)\, ds$$

Let $\ell = k-j$ so that $j=k-\ell$

$$\sum_{\ell=-\infty}^{\infty} \sum_{k=0}^{P-1} \eta\big(a(k)\big) \overline{\eta\big(b(k-\ell)\big)} \int_{-\infty}^{\infty} f_0(s)\, f_0\big(s-(\tau-\Delta\ell)\big)\, ds$$

$$\sum_{\ell=-\infty}^{\infty} \theta(a,b)(\ell)\; R\,(f_0,f_0)(\tau-\Delta\ell)$$

E.4 POWER SPECTRAL DENSITY OF FOUR-STATE ENCODED SIGNAL

Our objective in this section is to compute the power spectral density of a four-state encoded signal. We do this in the series of results that follow.

E.4-1 POWER SPECTRAL DENSITY OF ENCODED BASE BAND SIGNAL

Result - Let a be a four-state sequence of period P

$$\text{Let } f(t) = \sum_{k=-\infty}^{\infty} \eta\big(a(k)\big)\; \delta\,(t-\Delta k)$$

where $\eta(0) = 1 \qquad \eta(1) = -1 \qquad \eta(\beta) = i \qquad \eta(\beta^2) = -i$

The power density spectrum of the baseband signal f is given by

$$S(f)(\omega) = \frac{2}{(\Delta P)^2} \sum_{j=-\infty}^{\infty} F(\theta)(j)\ \delta\left(\omega - \frac{2\pi j}{\Delta P}\right)$$

Where θ is the auto-correlation function of the binary sequence a i.e.

$$\theta(a,a)(\tau) = \sum_{i=0}^{P-1} \eta\big(a(i)\big)\ \overline{\eta\big(a(i-\tau)\big)}$$ and $F(\theta)$ is the digital

fourier transform of the complex valued periodic sequence θ

$$F(\theta)(j) = \sum_{k=0}^{P-1} \theta(a,a)(k)\ e^{\frac{-2\pi i j k}{P}}$$

Proof: To compute the power density spectrum $S(f)$ of f we use the Weiner-Khinchin Theorem which states that $S(f)$ is $\sqrt{\frac{2}{\pi}}$ times the auto-correlation function of f. The auto-correlation function of f is given by

$$\psi(f)(\tau) = \frac{1}{\Delta P} \sum_{n=-\infty}^{\infty} \theta(a,a)(k)\ \delta\ (\tau - \Delta \cdot n)$$ and hence we have

$S(f)(\omega)$

$$\sqrt{\frac{2}{\pi}}\; F\big(\psi(f)\big)\omega$$

$$\sqrt{\frac{2}{\pi}}\; F\left[\frac{1}{\Delta P} \sum_{k=-\infty}^{\infty} \theta(a,a)(k)\; \delta\; (\omega - k\Delta)\right]$$

$$\left(\sqrt{\frac{2}{\pi}}\right)\left(\frac{1}{\Delta P}\right)\left(\frac{2\pi}{\Delta P}\right)\; F\left[\sum_{k=0}^{P-1} \theta(a,a)(k)\delta\; (\omega - k\Delta)\right] \sum_{j=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi j}{\Delta P}\right)$$

$$\frac{2}{\Delta^2 P^2} \sum_{k=0}^{P-1} \theta(a,a)(k)\; e^{\frac{-2\pi i jk}{P}} \sum_{j=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi j}{\Delta P}\right)$$

$$\frac{2}{(\Delta P)^2} \sum_{j=-\infty}^{\infty} F(\theta)(j)\; \delta\; \left(\omega - \frac{2\pi j}{\Delta P}\right)$$

Corollary: Let $g(t) = \sum_{k=-\infty}^{\infty} \eta\big(a(k)\big)\; \delta\; (t - \Delta k)$

and $f(t) = \sum_{k=-\infty}^{\infty} \eta\big(a(k)\big)\; f_0(t - \Delta k)$

Then the power spectral density $S(f)$ of f is given by

$S(f)(\omega) = \pi\; S(g)(\omega) \cdot E(f_0)(\omega)$

where $E(f_0)$ is the energy spectral density of f_0

Proof: f is the convolution of f_0 and g .

E.4-2 <u>POWER SPECTRAL DENSITY OF CARRIER ENCODED WITH A FOUR-STATE SEQUENCE</u>

We may now compute the power density spectrum of a carrier encoded with a four-state sequence.

Result: The power density spectrum $S(f)$ of the encoded signal

$$s(t) = \sum_{k=-\infty}^{\infty} \cos\left(\omega_0 t + \gamma\big(a(k)\big)\right) f_0(t - \Delta k) = \mathrm{Re}\left\{f(t) e^{i\omega_0 t}\right\}$$

where $f(t) = \sum_{k=-\infty}^{\infty} \eta\big(a(k)\big)\, f_0(t-\Delta k)$ is given by

$S(s)(\omega)=$

$$\frac{\pi}{2}\ \frac{1}{(\Delta P)^2} \left[E(f_0)(\omega-\omega_0) \cdot \sum_{j=-\infty}^{\infty} F\big(\theta(a)\big)(j)\ \delta\left((\omega-\omega_0) - \frac{2\pi j}{\Delta P}\right)\right.$$

$$\left. + \quad E(f_0)(\omega + \omega_0) \cdot \sum_{j=-\infty}^{\infty} F\big(\theta(a)\big)(j)\ \delta\left((\omega+ \omega_0) - \frac{2\pi j}{\Delta P}\right) \right]$$

Proof: We compute the power density spectrum of the encoded signal s by taking the fourier transform of its auto-correlation function R(s) which was shown to be

$$\mathrm{Re}\left\{ e^{i\omega_0\tau}\, R(f)(\tau) \right\}$$

$$S(s)(\omega)$$

$$\sqrt{\frac{2}{\pi}}\; F\big(R(s)\big)(\omega) \qquad \text{Weiner Khinchin Theorem}$$

$$\left(\sqrt{\frac{2}{\pi}}\right)\left(\frac{1}{2}\right) F\left[\mathrm{Re}\left\{ e^{i\omega_0\tau}\, R(f)(\tau) \right\}\right](\omega) \qquad \text{previous result for } R(s)$$

$$\left(\sqrt{\frac{2}{\pi}}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left[F\left(e^{i\omega_0\tau}\, R(f)(\tau)\right)(\omega) + \overline{F\left(e^{i\omega_0\tau}\, R(f)(\tau)\right)(-\omega)}\right]$$

$$\left(\sqrt{\frac{2}{\pi}}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left[F\big(R(f)(\tau)\big)(\omega-\omega_0) + F\big(R(f)(\tau)\big)(\omega+\omega_0)\right]$$

$$\left(\sqrt{\frac{2}{\pi}}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\sqrt{\frac{\pi}{2}}\right)\left[S(f)(\omega-\omega_0) + S(f)(\omega+\omega_0)\right] \qquad \text{Weiner Khinchin}$$

$$\left(\frac{\pi}{4}\right)\left[E(f_0)(\omega-\omega_0)\cdot S(g)(\omega-\omega_0) + E(f_0)(\omega+\omega_0)\cdot S(g)(\omega+\omega_0)\right]$$

by previous corollary where $g(t) = \sum_{k=-\infty}^{\infty} \eta\big(a(k)\big)\delta\,(t-\Delta k)$

The result now follows from the previous computation.

Corollary: If f_0 is the square pulse such that

$f_0(t) = 1 \quad 0<t<\Delta \quad$ and $f_0(t) = 0 \quad$ otherwise then

$S(f)(\omega) =$

$$\frac{1}{2P^2}\left[\frac{\sin^2\frac{(\omega-\omega_0)\Delta}{2}}{\left(\frac{(\omega-\omega_0)\Delta}{2}\right)^2}\sum_{j=-\infty}^{\infty} F(\theta)(j)\,\delta\left((\omega-\omega_0)-\frac{2\pi j}{\Delta P}\right)+\right.$$

$$\left.\frac{\sin^2\frac{(\omega+\omega_0)\Delta}{2}}{\left(\frac{(\omega+\omega_0)\Delta}{2}\right)^2}\sum_{j=-\infty}^{\infty} F(\theta)(j)\,\delta\left((\omega+\omega_0)+\frac{2\pi j}{\Delta P}\right)\right]$$

This result follows from the fact that

$$E(f_0)(\omega) + \frac{\Delta^2}{\pi}\quad\frac{\sin^2\left(\frac{\omega\Delta}{2}\right)}{\left(\frac{\omega\Delta}{2}\right)^2}$$

We have shown through the preceding calculations that the power spectral density of the four-phase encoded signal consists of spectral lines at multiples of the code repetition frequency and modulated by a $(\sin^2 x)/x^2$ envelope. The amplitude of the spectral lines are determined by the digital fourier transform of the complex auto-correlation function of the encoding four-state sequence. In what follows, we compute this digital fourier transform for a maximal four-state sequence and then completely determine the power spectral density of a carrier encoded with a four-state maximal PN sequence.

Result: Let a be a maximal four-state PN sequence of period $P = 4^n - 1$. We have shown that the auto-correlation function of a is given by:

$$\theta(a)(0) = P + 0j$$

$$\theta(a)(P/3) = -1 \pm \left(\frac{P+1}{2}\right) j$$

$$\theta(a)(2P/3) = -1 \mp \left(\frac{P+1}{2}\right) j$$

$$\theta(a)(\tau) = -1 + 0j \quad \text{Otherwise}$$

Let $F\left(\theta(a)\right)$ be the digital fourier transform of the correlation sequence $\theta(a)$ where

$$F\left(\theta(a)\right)(j) = \sum_{k=0}^{P-1} \theta(a)(k) e^{\frac{-2\pi i j k}{P}} .$$ Then

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$$F\,\theta(a)\,(j)\begin{cases} = (P+1)\left(1 \pm \dfrac{\sqrt{3}}{2}\right) & \text{for } j \equiv 1 \text{ modulo } 3 \\ = (P+1)\left(1 \mp \dfrac{\sqrt{3}}{2}\right) & \text{for } j \equiv 2 \text{ modulo } 3 \\ = (P+1) \quad \text{for } j \equiv 0 \text{ modulo } 3;\ j \not\equiv 0 \text{ modulo } P & \\ = 1 & \text{for } j \equiv 0 \text{ modulo } P \end{cases}$$

Proof: $F(\theta)(j)$

$$\sum_{k=0}^{P-1} \theta(a)(k)e^{\frac{-2\pi ijk}{P}}$$

$$\sum_{k=0}^{P-1} \mathrm{Re}\left(\theta(a)(k)\right) e^{\frac{-2\pi ijk}{P}} + i \sum_{k=0}^{P-1} \mathrm{Im}\,\theta(a)(k)e^{\frac{-2\pi ijk}{P}}$$

We now evaluate each of the above terms separately by substituting for the values of the sequence $\theta(a)$

$$\sum_{k=0}^{P-1} \mathrm{Re}\,\theta(a)e^{\frac{-2\pi ijk}{P}}$$

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$$P - \sum_{k=1}^{P-1} (\omega^j)^k \qquad \omega = e^{\frac{-2\pi i}{P}}$$

$$= \begin{cases} P+1 & \text{for } j \not\equiv 0 \text{ modulo } P \\ 1 & \text{for } j \equiv 0 \text{ modulo } P \end{cases}$$

$$i \sum_{k=0}^{P-1} \operatorname{Im} \theta(a)(k) e^{\frac{-2\pi ijk}{P}}$$

$$\pm \left(\frac{P+1}{2}\right) i\, e^{\frac{-2\pi ijk}{3}} \mp \left(\frac{P+1}{2}\right) i\, e^{\frac{2\pi ij}{3}}$$

$$(P+1)\left[\pm \sin \frac{2\pi}{3} j\right] = \begin{cases} 0 & \text{for } j \equiv 0 \text{ modulo } 3 \\ \pm \frac{\sqrt{3}}{2} & \text{for } j \equiv 1 \text{ modulo } 3 \\ \mp \frac{\sqrt{3}}{2} & \text{for } j \equiv 2 \text{ modulo } 3 \end{cases}$$

The above sum thus becomes

1 for $j \equiv 0$ modulo P

P+1 for $j \equiv 0$ modulo 3; $j \not\equiv 0$ modulo P

$$(P+1)\left(1 \pm \frac{\sqrt{3}}{2}\right) \qquad \text{for } j \equiv 1 \text{ modulo } 3$$

$$(P+1)\left(1 \mp \frac{\sqrt{3}}{2}\right) \qquad \text{for } j \equiv 2 \text{ modulo } 3$$

The spectral lines of binary and four-state sequences are illustrated in Figure E-3.

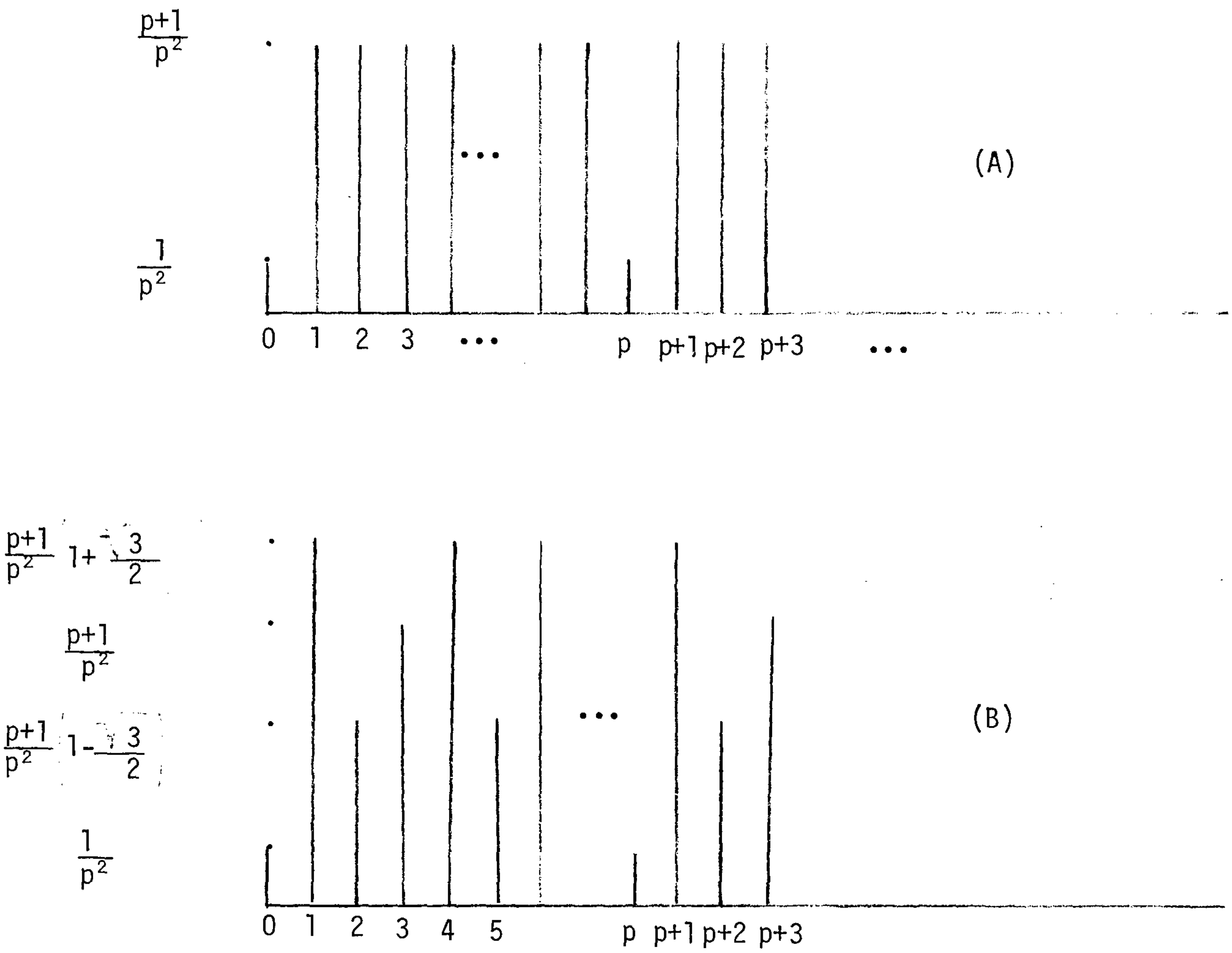

Figure E-3. Spectral lines of PN Sequences (A) Binary (B) Four-State